Ratio Mathematica Volume 48, 2023

On Ideals in Partially Ordered Ternary Semigroups

Dattatray Shinde*
Machchhindra Gophane †

Abstract

The concept of ideals has been extensively studied through various algebraic structures such as near-rings, involution rings, regular rings, gamma-rings, semigroups, ordered semigroups, ternary semigroups and ordered ternary semigroups. In this article, we have investigated some intriguing properties of ideals, bi-ideals and pseudo symmetric ideals in partially ordered ternary semigroup T. We establish conditions for $(UVW]$ to be an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. We also prove that the collection $\mathcal{P}$ of all pseudo symmetric ideals of T is a Brouwerian lattice. Also we define a pseudo symmetric partially ordered ternary semigroup.

Keywords: partially ordered ternary semigroup; bi-ideal; pseudo symmetric ideal; brouwerian lattice.
2020 AMS subject classifications: 20M12; 20N99; 06F99.[1]

*Department of Mathematics, Shivaji University, Kolhapur-416004, Maharashtra, India; shindednmaths@gmail.com.
†Department of Mathematics, Shivaji University, Kolhapur-416004, Maharashtra, India; mtg_maths@unishivaji.ac.in.
[1]Received on July 06, 2023. Accepted on November 12, 2023. Published on December 10, 2023. DOI: 10.23755/rm.v48i0.1288. ISSN: 1592-7415. eISSN: 2282-8214.

1 Introduction

Algebraic structures play a significant role in development of abstract algebra in particular and mathematics in general. Algebraic structure has a wide variety of applications in various fields such as theoretical physics, computing science, information technology, control engineering, topological spaces and many more. The study of different algebraic structures has attracted the attention of many researchers.

The concept of a semigroup in particular is a very simple mathematical structure. It plays incredible role in the evolution of abstract algebra. Many applications of semigroup structure can be found in many disciplines such as physics, combinatorics, coding theory, automata theory and so on. In probability theory, the semigroups are associated with their relationship to the Markov process. A semigroup is an algebraic structure consisting of a non-empty set together with an associative binary operation. A variety of studies have been conducted on the generalizations to the concept of semigroups viz. ordered semigroups, ternary semigroups and ordered ternary semigroups. Following this, researchers have expanded the research in a myriad of ways. N. Kehayopulu developed the idea of ordered semigroups and invented the concept of ordered bi-ideals in an ordered semigroup. He primarily devoted his time to study ordered ideals, ordered quasi-ideals and ordered bi-ideals in an ordered semigroup. In addition, he proposed the idea of regular ordered semigroups.

There is an extensive research literature that deals with the ternary operation. Firstly, ternary algebraic operations were introduced during the 19^{th} century by Cayley. The idea of ternary algebraic structures was developed by D. H. Lehmer in 1932. In [3], D. H. Lehmer studied the certain algebraic systems called as triplexes, which were discovered to be commutative ternary groups. Ideals have been extensively studied in various algebraic structures such as near-rings, involution rings, regular rings, gamma-rings, semigroups, ordered semigroups, ternary semigroups and ordered ternary semigroups. The ideal theory in n-ary semigroups and ternary semigroup was developed by Sioson F. M. [4] in 1965. The properties of quasi-ideals and bi-ideals in ternary semigroups studied by authors V. N. Dixit and S. Dewan [7].

A. Iampan in [1, 2] invented the notion of ordered ternary semigroup which is a generalization of the notion of ordered semigroups together with the notion of ternary semigroups and characterized the minimality and maximality of ordered lateral ideals in the ordered ternary semigroups. He also developed the theory of ordered ternary semigroups along the line of the theory of ordered semigroups. In 2014, V. Siva Rami Reddy et al. [9, 10] have developed the ideal theory of a partially ordered ternary semigroup. They also defined and studied the notions of prime ideals in partially ordered ternary semigroups. In [6], V. Jyothi et al.

introduced and studied the notion of pseudo symmetric ideals in partially ordered ternary semigroups.

In this article, we investigate several intriguing properties of ideals in partially ordered ternary semigroup T. We define the conditions under which $(UVW]$ is an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. Also, we prove that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Brouwerian lattice. Also we define a pseudo symmetric partially ordered ternary semigroup.

2 Preliminaries

In this section, we collect some definitions and fundamental results from [1, 2, 3, 6, 8], which we need for the development of this article.

A non-empty set T is said to be a ternary semigroup [3] if there exists a ternary operation [ ], from $T \times T \times T$ to T written as $(u, v, w) \longrightarrow [uvw]$, satisfies the law, $[u\ v\ w\ s\ t] = [[u\ v\ w]\ s\ t] = [u\ [v\ w\ s]\ t] = [u\ v\ [w\ s\ t]]$, for all $u, v, w, s, t \in T$.

Let U, V and W be three non-empty subsets of a ternary semigroup T, $[UVW] = \{[uvw] : u \in U, v \in V \text{ and } w \in W\}$. For easiness, we write, $[UVW]$ as UVW, $[uvw] = uvw$ and $[UUU] = U^3$.

A ternary semigroup T is called a partially ordered ternary semigroup [1] if there exist a partial ordering relation $\leq$ on T such that, $s \leq t \Rightarrow uvs \leq uvt, usv \leq utv, suv \leq tuv\ \forall s, t, u, v \in T$. In this article, let T be a partially ordered ternary semigroup, unless otherwise specified. For $U \subseteq T$, let $(U] = \{s \in T : s \leq u, \text{ for some } u \in U\}$. A partially ordered ternary semigroup T is said to be commutative if $uvw = wuv = vwu = vuw = wvu = uwv\ \forall u, v, w \in T$.

A left (resp., a right, a lateral) ideal of T is a non-empty subset I of T such that $TTI \subseteq I$ (resp., $ITT \subseteq I$, $TIT \subseteq I$) and $(I] = I$. A non-empty subset I of T is called a ideal of T if I is a left ideal, a right ideal and a lateral ideal of T. A bi-ideal of T is a non-empty subset U of T such that $UTUTU \subseteq U$ and $(U] = U$. An ideal P of T is said to be a pseudo symmetric ideal [6] of T provided $u, v, w \in T,\ uvw \in P \Rightarrow usvtw \in P\ \forall s, t \in T$.

Result 2.1 ([8]). *Let U, V and W be three non-empty subsets of T then the following hold,*
(1) $U \subseteq (U]$.
(2) $((U]] = (U]$.
(3) $(U](V](W] \subseteq (UVW]$.
(4) If $U \subseteq V$ *then* $(U] \subseteq (V]$.

3 Some Properties of Ideals

In this section, we study the various properties of ideals in a partially ordered ternary semigroup.

Theorem 3.1. *If U is a non-empty subset of T. Then $(TUT]$ is a two-sided ideal of T.*

Proof. As U is non-empty subset of T then $(TUT]$ is a non-empty subsets of T. Consider, $TT(TUT] = ((TTT)UT] \subseteq (TUT]$. Similarly, we can show that, $(TUT]TT \subseteq (TUT]$. Now, let $v \in (TUT]$ and $u \in T$ such that $u \leq v$. $v \in (TUT] \Rightarrow v \leq w$ for some $w \in TUT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in TUT \Rightarrow u \in (TUT]$. Hence $(TUT]$ is a two-sided ideal of T. □

Theorem 3.2. *If U and V are two non-empty subsets of T, then $(UVT]$ is a right ideal of T.*

Proof. Since U and V are two non-empty subsets of T then $(UVT]$ is a non-empty subsets of T. Consider, $(UVT]TT = (UV(TTT)] \subseteq (UVT]$. Now, let $v \in (UVT]$ and $u \in T$ such that $u \leq v$. $v \in (UVT] \Rightarrow v \leq w$ for some $w \in UVT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVT \Rightarrow u \in (UVT]$. Hence $(UVT]$ is a right ideal of T. □

Theorem 3.3. *If U is a left ideal and V is a non-empty subset of T, then $(UVT]$ is a two-sided ideal of T.*

Proof. As U, V are non-empty subset of T. So $(UVT]$ is a non-empty subset of T. Consider, $TT(UVT] = ((TTU)VT] \subseteq (UVT]$ (since U is a left ideal of T). Similarly, we can prove that, $(UVT]TT \subseteq (UVT]$. Now, let $v \in (UVT]$ and $u \in T$ such that $u \leq v$. $v \in (UVT] \Rightarrow v \leq w$ for some $w \in UVT$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVT \Rightarrow u \in (UVT]$. Hence $(UVT]$ is two-sided ideal of T. □

Remark 3.1. *Taking $V = T$ in particular, in Theorem 3.2 we get $(UTT]$ is right ideal of T. But the set $(UTT]$ is not a left and a lateral ideal of T. To establish this, consider the non-commutative partially ordered ternary semigroup T. If U is a left ideal of T then $(UTT]$ is a two-sided ideal of T.*

Theorem 3.4. *If U and V are two non-empty subsets of T, then $(TUV]$ is a left ideal of T.*

Proof. Similar to the proof of Theorem 3.2 □

Theorem 3.5. *If V is a right ideal of T and U is a non-empty subset of T, then $(TUV]$ is a two-sided ideal of T.*

Proof. Similar to the proof of Theorem 3.3 □

Remark 3.2. *Taking $U = T, V = U$ in particular, in Theorem 3.4 we get $(TTU]$ is a left ideal of T. But the set $(TTU]$ is not a right and a lateral ideal of T. To establish this, consider the non-commutative partially ordered ternary semigroup T. If U is a right ideal of T then $(TTU]$ is a two-sided ideal of T.*

Theorem 3.6. *If U is a left ideal and V is a right ideal of T then $(UTV]$ is two-sided ideal of T.*

Proof. As U, V are non-empty subset of T. So $(UTV]$ is a non-empty subset of T. Consider, $TT(UTV] = ((TTU)TV] \subseteq (UTV]$ (since U is a left ideal of T). Similarly, we can prove that, $(UTV]TT \subseteq (UTV]$. Now, let $v \in (UTV]$ and $u \in T$ such that $u \leq v$. $v \in (UTV] \Rightarrow v \leq w$ for some $w \in UTV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UTV \Rightarrow u \in (UTV]$. Hence $(UTV]$ is two-sided ideal of T. □

Corolary 3.1. *If U is a left ideal, W is a right ideal and V is a non-empty subset of T, then $(UVW]$ is two-sided ideal of T.*

Definition 3.1 ([5]). *A partially ordered ternary semigroup T is called regular partially ordered ternary semigroup if for each $u \in T$ there exists $v \in T$ such that $u \leq uvu$.*

Lemma 3.1 ([5]). *If T is a regular partially ordered ternary semigroup, then $U \subseteq (UTU]$ for any $U \subseteq T$.*

Theorem 3.7. *If U, V, W are three ideals of a regular partially ordered ternary semigroup T. Then $(UVW]$ is an ideal of T.*

Proof. Since U, V, W are three ideals of a regular partially ordered ternary semigroup T. By Lemma 3.1, $U \subseteq (UTU], V \subseteq (VTV]$ and $W \subseteq (WTW]$. Consider, $TT(UVW] = (T](T](UVW] = (TT(UVW)] = ((TTU)VW)] \subseteq (UVW]$. Similarly, we can show that, $(UVW]TT \subseteq (UVW]$. Now, consider $T(UVW]T = (T(UVW)T] \subseteq (T(UTU]V(WTW]T] \subseteq ((TUTUVWTWT]]$ $\subseteq (TUTUVWTWT] \subseteq ((TUT)UV(WTW)T] \subseteq (UUVW(TWT)] \subseteq$ $(UUVWW] \subseteq (UTVTW] \subseteq (U(TVT)W] \subseteq (UVW]$. Let $v \in (UVW]$ and $u \in T$ such that $u \leq v$. $v \in (UVW] \Rightarrow v \leq w$ for some $w \in UVW$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVW \Rightarrow u \in (UVW]$. Hence $(UVW]$ is an ideal of T. □

Note 3.1. *In general, if V is an ideal of U and U is an ideal of T. Then V is not an ideal of T.*

Theorem 3.8. *If U is an ideal of T and V is an ideal of U such that $V = V^3$. Then V is an ideal of T.*

Proof. As U is an ideal of T and V is an ideal of U such that $V = V^3$. Consider, $TTV = TT(VVV) \subseteq (TTU)VV \subseteq UVV \subseteq UUV \subseteq V$. Similarly, we can prove that, $VTT \subseteq V$. Now, consider $TVT = T(VVV)T \subseteq (TTU)VT \subseteq UVT = U(VVV)T \subseteq (UUV)VT \subseteq VVT = V(VVV)T \subseteq VV(UTT) \subseteq VVU \subseteq VUU \subseteq V$. Let $v \in V$ and $u \in T$ such that $u \leq v$. $v \in V \subseteq U, u \in T, u \leq v$ and U is an ideal of $T \Rightarrow u \in U$. Now, we have V is an ideal of U, $v \in V$ and $u \in U \subseteq T$ such that $u \leq v \Rightarrow u \in V$. Therefore V is an ideal of T. □

4 Results on Bi-ideals

In this section, we study the various properties of bi-ideals in a partially ordered ternary semigroup.

Theorem 4.1. *Let U be a left ideal and V be a bi-ideal of T. Then $(UVV]$ is a left ideal and a bi-ideal of T.*

Proof. As U, V are non-empty subsets of T. So $(UVV]$ is a non-empty subset of T. Consider $TT(UVV] = (T](T](UVV] = (TT(UVV)] = ((TTU)VV)] \subseteq (UVV]$. By definition of bi-ideal, consider $(UVV]T(UVV]T(UVV]$ $= (UVVTUVVTUVV] = (U(V(VTU)V(VTU)V)V] \subseteq (U(VTVTV)V]$ (since $(VTU) \subseteq (TTT) \subseteq T$) $\subseteq (UVV]$ (since V is a bi-ideal of T, $(VTVTV) \subseteq V$). Now, let $v \in (UVV]$ and $u \in T$ such that $u \leq v$. $v \in (UVV] \Rightarrow v \leq w$ for some $w \in UVV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVV \Rightarrow u \in (UVV]$. Hence $(UVV]$ is a left ideal and a bi-ideal of T. □

Theorem 4.2. *Let U be a right ideal and V be a bi-ideal of T. Then $(VVU]$ is a right ideal and a bi-ideal of T.*

Proof. Similar to the proof of Theorem 4.1 □

Theorem 4.3. *Let U be a lateral ideal and V be a bi-ideal of T. Then $(VUV]$ is a bi-ideal of T.*

Proof. As U, V are non-empty subsets of T, $(VUV]$ is a non-empty subset of T. By definition of bi-ideal, consider $(VUV]T(VUV]T(VUV] =$ $(VUVTVUVTVUV] = (V(U(VTV)U(VTV)U)V] \subseteq (V(UTUTU)V]$ (since

$(VTV) \subseteq (TTT) \subseteq T) \subseteq (VU(TUT)UV] \subseteq (VU(U)UV]$ (since U is a lateral ideal of $T, (TUT) \subseteq U) \subseteq (V(UUU)V] \subseteq (VUV]$ (since $(UUU) \subseteq U$). Now, let $v \in (VUV]$ and $u \in T$ such that $u \leq v$. $v \in (VUV] \Rightarrow v \leq w$ for some $w \in VUV$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in VUV \Rightarrow u \in (VUV]$. Hence $(VUV]$ is a bi-ideal of T. □

Note 4.1. *In general, if V is a bi-ideal of T and U is a bi-ideal of V. Then U is not a bi-ideal of T.*

Theorem 4.4. *If V is a bi-ideal of T and U is a bi-ideal of V such that $U = U^3$, then U is a bi-ideal of T.*

Proof. As V is a bi-ideal of T, $VTVTV \subseteq V$ and U is a bi-ideal of V, $UVUVU \subseteq U$. Consider $UTUTU = (UUU)TUT(UUU) = UU(UTUTU)UU \subseteq UU(VTVTV)UU \subseteq UUVUU = UUVU(UUU) \subseteq U(UVUVU)U \subseteq UUU = U$. Let $v \in U$ and $u \in T$ such that $u \leq v$. $v \in U \subseteq V, u \in T, u \leq v$ and V is a bi-ideal of $T \Rightarrow u \in V$. Now, we have U is a bi-ideal of V, $v \in U$ and $u \in V \subseteq T$ such that $u \leq v \Rightarrow u \in U$. Hence U is a bi-ideal of T. □

5 Properties of Pseudo Symmetric Ideals

In this section, we study the various properties of pseudo symmetric ideals in a partially ordered ternary semigroup.

Theorem 5.1. *If I is a pseudo symmetric ideal of T and U is a subsemigroup of T then $I \cap U$ is a pseudo symmetric ideal of U, provided $I \cap U \neq \emptyset$.*

Proof. Assume that $V = I \cap U \neq \emptyset$. Since $V \subseteq I$, it follows that $UUV \subseteq TTI \subseteq I$. Since $V \subseteq U$ and U is subsemigroup of T, we have $UUV \subseteq U$. Then $UUV \subseteq I \cap U \Rightarrow UUV \subseteq V$. Similarly, we can show that $VUU \subseteq V$ and $UVU \subseteq V$. Let $v \in V$ and $u \in U$ such that $u \leq v$. Since $v \in V \Rightarrow v \in I, v \in U$ and $u \leq v \Rightarrow u \in I$ and $u \in U \Rightarrow u \in V \Rightarrow (V] \subseteq V$. Obviously, $V \subseteq (V]$. This shows that $(V] = V$. Hence V is an ideal of U. Let $u, v, w \in U$ such that $uvw \in V \Rightarrow uvw \in I$ and $uvw \in U$. If $uvw \in I$ and I is pseudo symmetric ideal then for all $s, t \in U \subseteq T \Rightarrow usvtw \in I$. If $uvw \in U$ and U is subsemigroup of T then for all $s, t \in U \subseteq T$, consider $usvtw = (usv)tw \in UUU \subseteq U$. This shows that $usvtw \in V$. Hence $V = I \cap U$ is a pseudo symmetric ideal of U. □

Theorem 5.2. *If U is an ideal of T such that $T^3 \subseteq U$ and I is a pseudo symmetric ideal of T then $U \cap I$ is a pseudo symmetric ideal of T, provided $U \cap I \neq \emptyset$.*

Proof. Assume that, $C = U \cap I \neq \emptyset$. Therefore C is non-empty subset of T. Since, U is an ideal of T and I is a pseudo symmetric ideal of T then $U \cap I = C$

is an ideal of T. Now, let $u, v, w \in T, uvw \in C \Rightarrow uvw \in U$ and $uvw \in I$. If $uvw \in U$ and U is an ideal of T. For all $s, t \in T$, consider, $usvtw = (usv)tw \in UTT \subseteq U \Rightarrow usvtw \in U$. If $uvw \in I$ and I is an pseudo symmetric ideal of T then $usvtw \in I$, for all $s, t \in T$. Therefore $usvtw \in C$. Hence $C = U \cap I$ is a pseudo symmetric ideal of T. □

Remark 5.1. *Every ideal of commutative partially ordered ternary semigroup is a pseudo symmetric ideal.*

Remark 5.2. *Let I be a pseudo symmetric ideal of partial ordered ternary semigroup T, then $(I]$ is a pseudo symmetric ideal of T.*

Theorem 5.3. *Let U, V, W be three non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVW]$ is a pseudo symmetric ideal of T, if one of the following condition holds in T.*
(1) U is a left or a right or a lateral ideal of T.
(2) V is a left or a right or a lateral ideal of T.
(3) W is a left or a right or a lateral ideal of T.

Proof. Since U, V, W are three non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVW]$ is also non-empty subsets of T. Here we give the proof of statement (1) only. Similarly we can write for (2) and (3). If U is a left or a right or a lateral ideal of T then we prove that $(UVW]$ is a pseudo symmetric ideal of T. Firstly we prove $(UVW]$ is an ideal of T. Consider, $TT(UVW] = (T](T](UVW] = (TT(UVW)] = ((TTU)VW)] \subseteq (UVW]$. Similarly, we can show that, $(UVW]TT \subseteq (UVW]$ and $T(UVW]T \subseteq (UVW]$. Now, let $v \in (UVW]$ and $u \in T$ such that $u \leq v$. $v \in (UVW] \Rightarrow v \leq w$ for some $w \in UVW$. Therefore $u \leq v, v \leq w \Rightarrow u \leq w \Rightarrow u \leq w, u \in T, w \in UVW \Rightarrow u \in (UVW]$. This shows that $(UVW]$ is an ideal of T. By Remark 5.1, $(UVW]$ is a pseudo symmetric ideal of T. □

Corolary 5.1. *Let U be any non-empty subset of a commutative partially ordered ternary semigroup T. Then $(TUT], (TTU]$ and $(UTT]$ are pseudo symmetric ideals of T.*

Corolary 5.2. *Let U and V be any two non-empty subsets of a commutative partially ordered ternary semigroup T then $(UVT], (UTV]$ and $(TUV]$ are pseudo symmetric ideals of T.*

Definition 5.1. *A partially ordered ternary semigroup T is said to be pseudo symmetric if every ideal of T is pseudo symmetric ideal of T.*

Remark 5.3. *Every commutative partially ordered ternary semigroup is a pseudo symmetric partially ordered ternary semigroup.*

Lemma 5.1. *Let T be a partially ordered ternary semigroup such that $T = T^3$, then T is a pseudo symmetric partially ordered ternary semigroup.*

Proof. Let I be any ideal in T and let $u, v, w \in T$ such that $uvw \in I$. Since $T = T^3$, consider $vwu = (vwu)(vwu)(vwu) = vw(uvw)(uvw)u \in I \Rightarrow vwu \in I$. Similarly, $wuv = (wuv)(wuv)(wuv) = w(uvw)(uvw)uv \in I \Rightarrow wuv \in I$. Now, if $s, t \in T$ then $usvtw = (usvtw)(usvtw)(usvtw) = usvt[wu(svt)(wus)v]tw \in I \Rightarrow usvtw \in I$. Therefore I is a pseudo symmetric ideal of T. Hence T is a pseudo symmetric partially ordered ternary semigroup. □

Proposition 5.1. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ forms a poset with respect to the inclusion of sets.*

Definition 5.2 ([8]). *A family Ω of subsets of a poset P is a Moore family if it satisfies the following conditions,*
(1) $P \in \Omega$.
(2) Ω closed with respect to arbitrary intersection.

Theorem 5.4. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ forms a Moore family of subsets of T.*

Proof. As T itself is a pseudo symmetric ideal, then $T \in \mathcal{P}$. We know that, intersection of an arbitrary collection of pseudo symmetric ideals of T is a pseudo symmetric ideal of T. Therefore $\mathcal{P}$ is closed with respect to arbitrary intersection. Thus $\mathcal{P}$ forms a Moore family of subsets of T. □

Lemma 5.2. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a lattice under the operations of intersection and union.*

Proof. Since, $\sup\{U_i, V_j\} = U_i \cup V_j$ and $\inf\{U_i, V_j\} = U_i \cap V_j$ exists for all $U_i, V_j \in \mathcal{P}$. Therefore $\mathcal{P}$ is a lattice. □

Lemma 5.3. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a complete lattice under the binary operations of intersection and union.*

Proof. Let H be the subset of $\mathcal{P}$. Since, $\bigwedge H = \bigcap_i I_i$ and $\bigvee H = \bigcup_i I_i$ exist for all $I_i \in H$. Therefore $\bigwedge H$ and $\bigvee H$ exists for any subset H of $\mathcal{P}$. Hence $\mathcal{P}$ is a complete lattice. □

Theorem 5.5. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a distributive lattice under the binary operations of intersection and union.*

Remark 5.4. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a modular lattice under the binary operations of intersection and union.*

Theorem 5.6. *Let $\mathcal{P}$ be a family of all pseudo symmetric ideals of T, then $\mathcal{P}$ is a Brouwerian lattice.*

Proof. Let U and V be any two pseudo symmetric ideals of T such that $U \cap V \neq \emptyset$. Consider the family of ideals of T, $K = \{I \in \mathcal{P} : I \cap U \subseteq V\}$. Then by Zorn's lemma there exist a maximal element say M in K. Select $I \in \mathcal{P}$ such that $I \cap U \subseteq V$. To show that M is the greatest element of K. If $M = T$ then the result follows. Assume that, $M \neq T$. Then we have, $U \cap (I \cup M) = (U \cap I) \cup (U \cap M) \subseteq V \cup V \subseteq V$. Thus $U \cap (I \cup M) \subseteq V$. As M is maximal in $\mathcal{P}$, we have $I \cup M = M$. This gives $I \subseteq M$. Thus the set K has the greatest element M. Hence $\mathcal{P}$ is a Brouwerian lattice. □

6 Conclusions

This paper is a continuation of the study of ideals in a partially ordered ternary semigroup. We have studied the several intriguing properties of ideals in partially ordered ternary semigroup T. We have established conditions for $(UVW]$ to be an ideal, bi-ideal and pseudo symmetric ideal of T, where U, V, W are non-empty subsets of T. We have proved that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Moore family of subsets of T. Furthermore, we have proved that the family $\mathcal{P}$ of all pseudo symmetric ideals of T forms a Brouwerian lattice. We further extend this study to other ternary algebraic structures in the future.

Acknowledgments

The authors express gratitude to the anonymous reviewers for their invaluable feedback and suggestions, which have significantly enhanced the paper's overall quality and clarity of presentation.

References

[1] A. Iampan, Characterizing the minimality and maximality of ordered lateral ideals in ordered ternary semigroups, *J. Korean Math. Soc.*, 46: 775–784, 2009.

[2] A. Iampan, On ordered ideal extensions of ordered ternary semigroups, *Lobachevskii J. Math.*, 31 (1) : 13–17, 2010.

[3] D. H. Lehmer, A ternary analogue of abelian groups, *Amer. J. Math.*, 54: 329–338, 1932.

[4] F. M. Sioson, Ideal theory in ternary semigroups, *Math. Japan*, 10: 63 – 84, 1965.

[5] P. Pornsurat and B. Pibaljommee, Regularities in ordered ternary semigroups, *Quasigroups Relat. Syst.*, 27 (1): 107–118, 2019.

[6] V. Jyothi, Y. Sarala and D. Madhusudhana Rao, Semipseudo symmetric ideals in partially ordered ternary semigroups, *Inter. J. Innovative Research and Development*, 3 (4): 386–393, 2014.

[7] V. N. Dixit and S. Dewan, A note on quasi and bi-ideals in ternary semigroups, *Int. J. Math. Math. Sci.*, 18 (3): 501 – 508, 1995.

[8] V. R. Daddi and Y. S. Pawar, A study of ternary semigroups and ternary semirings, Thesis, 2013.

[9] V. Siva Rami Reddy, V. Sambasiva Rao, A. Anjaneyulu and A. Gangadhara Rao, Ideals in partially ordered ternary semigroups, *Inter. J. Math. Sci., Technology and Humanities*, 110: 1177–1194, 2014.

[10] V. Siva Rami Reddy, V. Sambasiva Rao, A. Anjaneyulu and A. Gangadhara Rao, Complete prime po ideals and prime po ideals in po ternary semigroups, *Inter. Research J. Pure Algebra*, 4 (3): 457–470, 2014.

Quasigroups and Related Systems **32** (2024), 41 − 48

https://doi.org/10.56415/qrs.v32.04

On pseudo-ideals in partially ordered ternary semigroups

Machchhindra Gophane and Dattatray Shinde

Abstract. We study the properties of different types of pseudo-ideals of a partially ordered ternary semigroup and prove that the space of all strongly irreducible pseudo-ideals of a partially ordered ternary semigroup is a compact space.

1. Introduction

In [2], Hewitt and Zuckerman specified the method of construction of ternary semigroups from binary and specified various connections between such semigroups. Ternary semigroups are a special case of n-ary semigroups. So many results on ternary semigroups has an analogous version for n-ary semigroups. F.M. Sioson [5] proved some results on ideals in ternary semigroups. In [1], W.A. Dudek and I.M. Groździńska characterized some classes of regular ternary semigroups by ideals can be deduced from general results proved for n-ary semigroups. The notion of prime, semiprime and strongly prime bi-ideals in ternary semigroups was introduced by M. Shabir and M. Bano in [4]. The concept of ordered ternary semigroups was developed by A. Iampan in [3].

Our aim of this article is to introduce the concepts of prime pseudo-ideals and irreducible pseudo-ideals in a partially ordered ternary semigroup and to study their properties. We also prove that the space of all strongly irreducible pseudo-ideals of a partially ordered ternary semigroup is a compact space.

2020 Mathematics Subject Classification: 20M12, 20N99, 06F99.
Keywords: partially ordered ternary semigroup, pseudo-ideal, prime pseudo-ideal, irreducible pseudo-ideal.

2. Preliminaries

A non-empty set T with a ternary operation $[\,]: T \times T \times T \longrightarrow T$ is called a *ternary semigroup* if $[\,]$ satisfies the associative law, $[abcde] = [[abc]de] = [a[bcd]e] = [ab[cde]]$, for all $a, b, c, d, e \in T$.

For non-empty subsets X, Y and Z of a ternary semigroup T, $[XYZ] = \{[xyz] : x \in X, y \in Y \text{ and } z \in Z\}$. We write, $[XYZ]$ as XYZ, $[xyz] = xyz$ and $[XXX] = X^3$.

A ternary semigroup T is said to be a *partially ordered ternary semigroup* if there exist a partially ordered relation $\leqslant$ on T such that, $a \leqslant b \Rightarrow xya \leqslant xyb$, $xay \leqslant xby$, $axy \leqslant bxy$ for all $a, b, x, y \in T$. In this article, we write T for a partially ordered ternary semigroup, unless otherwise specified.

An element $e \in T$ is said to be an *identity element* of T if $exx = xxe = xex = x$ for all $x \in T$.

The set $\{t \in T : t \leqslant x, \text{for some} x \in X\}$ is denoted by $(X]$. A non-empty subset X of T is said to be a *partially ordered ternary subsemigroup* of T, if $[XXX] \subseteq X$ and $(X] = X$. A non-empty subset I of T is said to be a *partially ordered left* (respectively, *right, lateral*) *ideal* of T if $TTI \subseteq I$ (respectively, $ITT \subseteq I$, $TIT \subseteq I$) and $(I] = I$.

A non-empty subset I of T is said to be *ideal* of T if it is a left ideal, a right ideal and a lateral ideal of T.

A partially ordered ternary subsemigroup I of T is called a *left* (respectively a *right*, a *lateral*) pseudo-ideal of T if $[xxxxI] \subseteq I$ (respectively, $[Ixxxx] \subseteq I$, $[xxIxx] \subseteq I$) for all $x \in T$. A pseudo-ideal I of T is said to be *proper pseudo-ideal* of T if it differs from T.

A non-empty subset I of T is said to be *two sided pseudo-ideal* of T, if it is both left and right pseudo-ideal of T. A non-empty subset I of T is said to be *pseudo-ideal* of T, if I is a left, a right and a lateral pseudo-ideal of T. Note that, the non-empty intersection of an arbitrary collection of pseudo-ideals of T is a pseudo-ideal of T.

Example 2.1. Let $\mathbb{N}$ be the set of all natural numbers. Define ternary operation $[\,]$ on $\mathbb{N}$ by $[xyz] = xyz$ for all $x, y, z \in \mathbb{N}$, where $\cdot$ is a usual multiplication and a usual partial ordering relation $\leqslant$ on $\mathbb{N}$. Then $\mathbb{N}$ is a partially ordered ternary semigroup and $I = 3\mathbb{N}$ is a pseudo-ideal of $\mathbb{N}$.

Definition 2.2. A proper pseudo-ideal I of a partially ordered ternary semigroup T is called

(i) *prime pseudo-ideal* of T if $XYZ \subseteq I$ implies $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$ for any pseudo-ideals X, Y, Z of T, (*ii*) *strongly prime pseudo-ideal* of T if $XYZ \cap YZX \cap ZXY \subseteq I$ implies $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$ for any pseudo-ideals X, Y, Z of T,

(*iii*) *semiprime pseudo-ideal* of T if X is a pseudo-ideal of T and $X^n \subseteq I$ implies $X \subseteq I$ for some odd natural number n.

3. Main Results

Definition 3.1. A proper pseudo-ideal I of T is said to be *irreducible* (respectively, *strongly irreducible*) pseudo-ideal of T if $X \cap Y \cap Z = I$ (respectively $X \cap Y \cap Z \subseteq I$) implies $X = I$ or $Y = I$ or $Z = I$ (respectively $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$) for all pseudo-ideals X, Y, Z of T.

Remark 3.2. Every strongly irreducible pseudo-ideal of T is an irreducible pseudo-ideal of T but converse is not true in general.

Theorem 3.3. *Let X be a proper pseudo-ideal of T. For any $t(\neq 0) \in T \backslash X$ there exists an irreducible pseudo-ideal I of T such that $X \subseteq I$ and $t \notin I$.*

Proof. Let $\mathcal{I} = \{X_\alpha : X_\alpha$ is a pseudo-ideal of $T, X \subseteq X_\alpha, t \notin X_\alpha\}$, where $\alpha \in \Delta$ is any indexing set. As X is a pseudo-ideal of T and $t \notin X$, we have $X \in \mathcal{I}$, so $\mathcal{I} \neq \emptyset$. Evidently $\mathcal{I}$ is partially ordered set under the inclusion of sets. If $\{X_i : i \in \Delta\}$ is a totally ordered subset (chain) of $\mathcal{I}$ then $\bigcup_{i \in \Delta} X_i$ is a pseudo-ideal of T containing X and $t \notin \bigcup_{i \in \Delta} X_i$. Therefore $\bigcup_{i \in \Delta} X_i$ is an upper bound of $\{X_i : i \in \Delta\}$. Thus every chain in $\mathcal{I}$ has an upper bound in $\mathcal{I}$. Hence by Zorn's Lemma, there exists a maximal element say I in the collection $\mathcal{I}$. This shows that I is a pseudo-ideal of T such that $X \subseteq I$ and $t \notin I$.

Now we show that I is an irreducible pseudo-ideal of T. Let I_1, I_2 and I_3 be any three pseudo-ideals of T such that $I = I_1 \cap I_2 \cap I_3$ then $I \subseteq I_1, I \subseteq I_2$ and $I \subseteq I_3$. If I_1, I_2 and I_3 properly contain I, then according to hypothesis $t \in I_1, t \in I_2$ and $t \in I_3$. Thus $t \in I_1 \cap I_2 \cap I_3 = I$. Which contradicts to the fact that $t \notin I$. Therefore either $I = I_1$ or $I = I_2$ or $I = I_3$. Hence I is an irreducible. □

Theorem 3.4. *Any proper pseudo-ideal of T is the intersection of all irreducible pseudo-ideals containing it.*

Proof. Let X be the any proper pseudo-ideal of T and $\{X_i : i \in \Delta\}$ be the family of all irreducible pseudo-ideals of T containing X. Then $X \subseteq \bigcap_{i\in\Delta} X_i$. If $X \subsetneq \bigcap_{i\in\Delta} X_i$ then there exists $t(\neq 0) \in \bigcap_{i\in\Delta} X_i$ such that $t \notin X$. This implies $t \in X_i\ \forall i \in \Delta$. Since $t \notin X$, then by Theorem 3.3, there exists an irreducible pseudo-ideal say Y of T containing X but not containing t. This is a contradiction to $t \in X_i\ \forall i \in \Delta$. Thus $\bigcap_{i\in\Delta} X_i \subseteq X$. Hence $X = \bigcap_{i\in\Delta} X_i$. □

Theorem 3.5. *Every strongly irreducible semiprime pseudo-ideal of T is a strongly prime pseudo-ideal of T.*

Proof. Let I be a strongly irreducible semiprime pseudo-ideal of T. If X, Y and Z are three pseudo-ideals of T such that $[XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. Then $(X\cap Y\cap Z)^3 = [(X\cap Y\cap Z)(X\cap Y\cap Z)(X\cap Y\cap Z)] \subseteq [XYZ]$. Similarly $(X \cap Y \cap Z)^3 \subseteq [YZX]$ and $(X \cap Y \cap Z)^3 \subseteq [ZXY]$. This proves that $(X\cap Y\cap Z)^3 \subseteq [XYZ]\cap[YZX]\cap[ZXY] \subseteq I$. Therefore $(X\cap Y\cap Z)^3 \subseteq I$. Since I is a semiprime pseudo-ideal, $(X \cap Y \cap Z) \subseteq I$. Also since I is a strongly irreducible pseudo-ideal of T. Therefore, by definition of strongly irreducible pseudo-ideal, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. Hence I is a strongly prime pseudo-ideal of T. □

Corollary 3.6. *Every strongly irreducible semiprime pseudo-ideal of T is prime pseudo-ideal of T.*

Definition 3.7. A pseudo-ideal X of partially ordered ternary semigroup T is called idempotent if $X^3 = X$.

Theorem 3.8. *The following assertions for a partially ordered ternary semigroup T with identity are equivalent.*

- (i) *Every pseudo-ideal of T is idempotent.*
- (ii) *For every three pseudo-ideals X, Y, Z of T,*

  $X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY]$.
- (iii) *Every proper pseudo-ideal of T is semiprime.*
- (iv) *Each proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which contain it.*

Proof. $(i) \Rightarrow (ii)$: Suppose that, every pseudo-ideal of T is idempotent. Let X, Y and Z be three pseudo-ideals of T. Then $X \cap Y \cap Z$ is a pseudo-ideal of T, so $X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq [XYZ]$. Similarly $X \cap Y \cap Z \subseteq [YZX]$ and $X \cap Y \cap Z \subseteq [ZXY]$. Therefore $X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY]$.

$(ii) \Rightarrow (i)$: Let X be a pseudo-ideal of T. Then from $(ii), X = X \cap X \cap X \subseteq [XXX] \cap [XXX] \cap [XXX] = [XXX] = X^3 \Rightarrow X \subseteq X^3$. As X be a pseudo-ideal of T, so $X^3 \subseteq X$. Thus $X^3 = X$. This shows that every pseudo-ideal of T is idempotent.

$(i) \Rightarrow (iii)$: Suppose that, every pseudo-ideal of T is idempotent. Let X be a proper pseudo-ideal of T. Let Y be a pseudo-ideal of T such that $Y^3 \subseteq X$, then by hypothesis $Y^3 = Y$. Thus $Y \subseteq X$. This shows that X is semiprime pseudo-ideal of T. Hence every pseudo-ideal of T is semiprime.

$(iii) \Rightarrow (iv)$: Suppose that each proper pseudo-ideal of T is semiprime. By Theorem 3.4, any proper pseudo-ideal X of T is the intersection of all irreducible pseudo-ideals of T containing it. By (iii), every proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which containing it.

$(iv) \Rightarrow (i)$: Suppose that each proper pseudo-ideal of T is the intersection of all irreducible semiprime pseudo-ideals of T which contain it. Let X be a pseudo-ideal of T. Therefore it is the intersection of all irreducible semiprime pseudo-ideals of T which contain it. Therefore X is a semiprime pseudo-ideal of T. As $X^3 \subseteq X^3 \Rightarrow X \subseteq X^3$ but $X^3 \subseteq X$ always. This shows that $X = X^3$. Hence every pseudo-ideal of T is idempotent. □

Theorem 3.9. *If every pseudo-ideal of T is strongly prime pseudo-ideal of T then each pseudo-ideal of T is idempotent.*

Proof. Suppose that, each pseudo-ideal of T is strongly prime, then each pseudo-ideal of T is semiprime. Thus by Theorem 3.8, every pseudo-ideal of T is idempotent. □

Theorem 3.10. *If every pseudo-ideal of T is idempotent and the set of pseudo-ideals of T is totally ordered under set inclusion then each pseudo-ideal of T is strongly prime pseudo-ideal of T.*

Proof. Suppose that every pseudo-ideal of T is idempotent and the set of pseudo-ideals of T is totally ordered under set inclusion. Let I, X, Y and Z be pseudo-ideals of T such that $[XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. As every pseudo-ideal of T is idempotent so, $X \cap Y \cap Z$ is idempotent. Then

$X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq [XYZ]$. Similarly $X \cap Y \cap Z \subseteq [YZX]$ and $X \cap Y \cap Z \subseteq [ZXY]$. Therefore $X \cap Y \cap Z \subseteq [XYZ] \cap [YZX] \cap [ZXY] \subseteq I$. As the set of all pseudo-ideal of T is totally ordered under set inclusion, therefore for pseudo-ideals X, Y, Z of T, we have the following six possibilities,

1) $X \subseteq Y \subseteq Z$, 2) $X \subseteq Z \subseteq Y$, 3) $Y \subseteq X \subseteq Z$
4) $Y \subseteq Z \subseteq X$, 5) $Z \subseteq X \subseteq Y$, 6) $Z \subseteq Y \subseteq X$.

In such cases, we have respectively,

1) $X \cap Y \cap Z = X$, 2) $X \cap Y \cap Z = X$, 3) $X \cap Y \cap Z = Y$,
4) $X \cap Y \cap Z = Y$, 5) $X \cap Y \cap Z = Z$, 6) $X \cap Y \cap Z = Z$.

Therefore $X \cap Y \cap Z = X$ or $X \cap Y \cap Z = Y$ or $X \cap Y \cap Z = Z$. Thus from $X \cap Y \cap Z \subseteq I$, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. This shows that I is a strongly prime pseudo-ideal of T. □

Theorem 3.11. *If the set of pseudo-ideals of T is totally ordered under set inclusion then every pseudo-ideal of T is idempotent if and only if each pseudo-ideal of T is prime.*

Proof. Suppose that every pseudo-ideal of T is idempotent. Let I, X, Y and Z be pseudo-ideals of T such that $XYZ \subseteq I$. As every pseudo-ideal of T is idempotent so, $X \cap Y \cap Z$ is idempotent. Then $X \cap Y \cap Z = (X \cap Y \cap Z)^3 = [(X \cap Y \cap Z)(X \cap Y \cap Z)(X \cap Y \cap Z)] \subseteq XYZ \subseteq I$. Therefore $X \cap Y \cap Z \subseteq I$. As in the proof of the Theorem 3.10 we get $X \cap Y \cap Z = X$ or $X \cap Y \cap Z = Y$ or $X \cap Y \cap Z = Z$. Thus from $X \cap Y \cap Z \subseteq I$, either $X \subseteq I$ or $Y \subseteq I$ or $Z \subseteq I$. This shows that I is a prime pseudo-ideal of T.

Conversely, suppose that every pseudo-ideal of T is a prime pseudo-ideal of T. Since the set of pseudo-ideals of T is totally ordered under set inclusion, therefore the concepts of primeness and strongly primeness coincide. Hence by Theorem 3.9, every pseudo-ideal of T is idempotent. □

Definition 3.12. An proper pseudo-ideal X of T is said to be *maximal pseudo-ideal* of T if X is not properly contained in any proper pseudo-ideal of T.

Theorem 3.13. *Every maximal pseudo-ideal X of T is irreducible pseudo-ideal of T.*

Proof. Let X be a maximal pseudo-ideal of T. Suppose X is not irreducible pseudo-ideal of T. i.e. for any three pseudo-ideals A, B and C of T such that $A \cap B \cap C = X \Rightarrow A \neq X, B \neq X$ and $C \neq X \Rightarrow X \subset A \subset T, X \subset$

$B \subset T, X \subset C \subset T$. Which is contradiction to X be a maximal pseudo-ideal of T. Hence X is an irreducible pseudo-ideal of T. $\square$

Definition 3.14. Let X be the non-empty subset of T. Then the intersection of all pseudo-ideals of T containing X is the smallest pseudo-ideal of T containing X. This pseudo-ideal of T is called the pseudo-ideal of T generated by X and it is denoted by $(X)_{pi}$. A pseudo-ideal I of T is said to be the principal pseudo-ideal generated by an element x if I is a pseudo-ideal generated by $\{x\}$ for some $x \in T$ and is denoted by $(x)_{pi}$.

Let $\mathfrak{A}$ be the set of all pseudo-ideals of T and $\mathfrak{B}$ be the set of all strongly irreducible pseudo-ideals of T. For each $X \in \mathfrak{A}$, we define $\Psi_X = \{Y \in \mathfrak{B} : X \nsubseteq Y\}$

Theorem 3.15. *The family,* $\mathfrak{J}(\mathfrak{B}) = \{\Psi_X : X \in \mathfrak{A}\}$ *forms a topology on the set* $\mathfrak{B}$.

Proof. (i) As $\{0\} \in \mathfrak{A}$, so $\Psi_{\{0\}} = \{Y \in \mathfrak{B} : \{0\} \nsubseteq Y\} = \emptyset$. Thus $\emptyset \in \mathfrak{J}(\mathfrak{B})$.

(ii) Since $T \in \mathfrak{A}$, we have $\Psi_T = \{Y \in \mathfrak{B} : T \nsubseteq Y\} = \mathfrak{B}$ because $\mathfrak{B}$ is the collection of all proper strongly irreducible pseudo-ideals of T. Thus $\mathfrak{B} \in \mathfrak{J}(\mathfrak{B})$.

(iii) Let $\Psi_{X_1}, \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$. We show that $\Psi_{X_1} \cap \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$. Let $Y \in \Psi_{X_1} \cap \Psi_{X_2}$ then $Y \in \mathfrak{B}$ such that $X_1 \nsubseteq Y$ and $X_2 \nsubseteq Y$. Suppose that $X_1 \cap X_2 \subseteq Y$. Now, we have $X_1 \cap X_2 \cap T = X_1 \cap X_2 \subseteq Y$. Since Y is a strongly irreducible pseudo-ideal of T, therefore either $X_1 \subseteq Y$ or $X_2 \subseteq Y$ or $T \subseteq Y$. But $T \nsubseteq Y$ (since Y is proper). Therefore $X_1 \subseteq Y$ or $X_2 \subseteq Y$, which is a contradiction. Hence $X_1 \cap X_2 \nsubseteq Y$. Therefore $Y \in \Psi_{X_1 \cap X_2}$. Thus $\Psi_{X_1} \cap \Psi_{X_2} \subseteq \Psi_{X_1 \cap X_2}$. On the other hand if $Y \in \Psi_{X_1 \cap X_2}$ then $Y \in \mathfrak{B}$ and $X_1 \cap X_2 \nsubseteq Y$. This implies that $X_1 \nsubseteq Y$ and $X_2 \nsubseteq Y$. Therefore $Y \in \Psi_{X_1}$ and $Y \in \Psi_{X_2} \Rightarrow Y \in \Psi_{X_1} \cap \Psi_{X_2}$. Hence $\Psi_{X_1 \cap X_2} \subseteq \Psi_{X_1} \cap \Psi_{X_2}$.This shows that $\Psi_{X_1} \cap \Psi_{X_2} = \Psi_{X_1 \cap X_2}$. Thus $\Psi_{X_1} \cap \Psi_{X_2} \in \mathfrak{J}(\mathfrak{B})$.

(iv) Let $\{X_\alpha\}_{\alpha \in \Delta}$ (where Δ is any indexing set.) be family of pseudo-ideals of T and $\{\Psi_{X_\alpha} : \alpha \in \Delta\} \subseteq \mathfrak{J}(\mathfrak{B})$. Then $\bigcup_{\alpha \in \Delta} \Psi_{X_\alpha} = \{Y \in \mathfrak{B} : X_\alpha \nsubseteq Y$ for some $\alpha \in \Delta\} = \{Y \in \mathfrak{B} : \left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi} \nsubseteq Y\} = \Phi_{\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi}} \in \mathfrak{J}(\mathfrak{B})$, where $\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)_{pi}$ is the pseudo-ideal of T generated by $\left(\bigcup_{\alpha \in \Delta} X_\alpha\right)$. Therefore from $(i), (ii), (iii)$ and (iv), we get the set $\mathfrak{J}(\mathfrak{B})$ forms a topology on $\mathfrak{B}$. $\square$

Theorem 3.16. *If T is partially ordered ternary semigroup with identity then $\mathfrak{B}$ is a compact space.*

Proof. Suppose that $\{\Psi_{X_k} : k \in \Delta\}$ is an open covering of $\mathfrak{B}$, where Δ is an indexing set. That is $\mathfrak{B} = \bigcup_{k\in\Delta} \Psi_{X_k}$. By Theorem 3.15, $\Psi_T = \mathfrak{B}$, therefore $\Psi_T = \bigcup_{k\in\Delta} \Psi_{X_k} \Rightarrow \Psi_T = \Psi_{\left(\bigcup_{k\in\Delta} X_k\right)_{pi}} \Rightarrow T = \left(\bigcup_{k\in\Delta} X_k\right)_{pi}$. As $e \in T, e \in \left(\bigcup_{k\in\Delta} X_k\right)_{pi}$. Hence $e \in (\bigcup_{i=1}^{n} X_i)_{pi} \Rightarrow T = (\bigcup_{i=1}^{n} X_i)_{pi} \Rightarrow \mathfrak{B} = \bigcup_{k=1}^{n} \Psi_{X_k}$. This shows that every open cover of $\mathfrak{B}$ has finite subcover. Hence $\mathfrak{B}$ is compact space. □

References

[1] **W.A. Dudek and I.M. Groździńska**, *On ideals in regular n-semigroups*, Matematići Bilten, (Skopje) **29/30** (1979-1980), 35 − 44.

[2] **E. Hewitt and H.S. Zuckerman**, *Ternary operations and semigroups*, Proc. Sympos. Wayne State Univ., Detroit (1968), 55 − 83.

[3] **A. Iampan**, *On characterizations of* (0−) *minimal and maximal ordered left* (*right*) *ideals in ordered ternary semigroups*, Global J. Pure Appl. Math., **4(1)** (2008), 91 − 100.

[4] **M. Shabir and M. Bano**, *Prime bi-ideals in partially ordered ternary semigroups*, Quasigroups Related Systems, **16** (2008), 239 − 256.

[5] **F.M. Sioson**, *Ideal theory in partially ordered ternary semigroups*, Math. Japan., **10** (1965), 63 − 84.

Received July 26, 2023

|
Department of Mathematics,
Shivaji University, Kolhapur- 416004,
Maharashtra (India)
E-mail: shindednmaths@gmail.com, mtg_maths@unishivaji.ac.in